PROBABILITY AND FOURIER DUALITY FOR AFFINE ITERATED FUNCTION SYSTEMS

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Abstract. Let d be a positive integer, and let μ be a finite measure on $\mathbb{R}^d$. In this paper we ask when it is possible to find a subset Λ in $\mathbb{R}^d$ such that the corresponding complex exponential functions e_λ indexed by Λ are orthogonal and total in $L^2(\mu)$. If this happens, we say that (μ, Λ) is a spectral pair. This is a Fourier duality, and the x-variable for the $L^2(\mu)$-functions is one side in the duality, while the points in Λ is the other. Stated this way, the framework is too wide, and we shall restrict attention to measures μ which come with an intrinsic scaling symmetry built in and specified by a finite and prescribed system of contractive affine mappings in $\mathbb{R}^d$; an affine iterated function system (IFS). This setting allows us to generate candidates for spectral pairs in such a way that the sets on both sides of the Fourier duality are generated by suitably chosen affine IFSs. For a given affine setup, we spell out the appropriate duality conditions that the two dual IFS-systems must have. Our condition is stated in terms of certain complex Hadamard matrices. Our main results give two ways of building higher dimensional spectral pairs from combinatorial algebra and spectral theory applied to lower dimensional systems.

Contents

1. Introduction

The use of traditional Fourier series has up to recently been restricted to the setting of Fourier duality between groups; in the abelian case [Rud62], between compact groups (such as tori) on the one side, each group coming with its Haar measure; and discrete abelian groups (such as lattices) on the other. However in dynamics and in other applications to computational mathematics, one is often faced with sets arising as attractors, highly non-linear, and coming equipped with equilibrium measures. This has led to attempts at adapting traditional Fourier tools to these non-linear and non-group settings. In this paper we address the Fourier duality question for affine iterated function systems. For some of

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the earlier literature we refer the reader to [JP92, JP93, JP94, JP95, JP98c, JP98a, DJ07b, DJ07c, DJ07a, DJ07d, DR07, DHPS08, Jor06, JKS07, OS05].

Iterated function systems (IFS) in $\mathbb{R}^d$ are natural generalizations of more familiar Cantor sets on the real line. Like their linear counterparts, they arise as limit sets X for recursively defined dynamical systems. While the functions used may be affine, the limit X itself will typically be a highly non-linear object, and will include complicated geometries. They arise in operator algebras and in representation theory; and they form models for "attractors" in dynamical systems arising in nature. For $d = 2$, the Sierpinski gasket is a notable example, and there is a variety of possibilities for $d > 2$ as well. Each affine IFS X possesses (normalized) invariant measures μ (see (2.6)), naturally associated with the system at hand (denote by X the support of the measure).

Question: When is there some *orthogonal basis* in the Hilbert space $L^2(X, \mu)$ of the form $\{e_\lambda \,|\, \lambda \in \Lambda\}$, where $\Lambda \subset \mathbb{R}^d$ and $e_\lambda(x) = \exp(2\pi i x \cdot \lambda)$?

Definition. The restricted class of IFSs μ for which a basis $\{e_\lambda\}_{\lambda\in\Lambda}$ can be found are called *spectral measures*, and the functions e_λ are said to form a *Fourier basis.* (μ, Λ) is then called a *spectral pair*.

If X is the middle-third Cantor set, Jorgensen and Pedersen [JP98b] proved that there is no such Fourier basis. Nonetheless, many spectral measures have since been found within various important classes of fractals; and their significance has been explored by many authors.

Affine IFSs often take the following form: Start with a d by d matrix R and a finite subset B in $\mathbb{R}^d$. Then consider the associated set of affine mappings in $\mathbb{R}^d$, τ_b of the form $\tau_b(x) = R^{-1}(x + b)$, where R is further assumed to be an expanding integer matrix and $b \in B \subset \mathbb{R}^d$.

In [DJ07c] we conjectured that the *spectral measures* arise precisely when there exists *duality pairing*, i.e., another system L such that the two define a *complex Hadamard matrix* H (see equation (1.1)), the order of the matrix H being the cardinality of B. We proved the conjecture in [DJ07c] when an additional assumption, called "reducibility", is placed on the triple (R, B, L).

Two approaches to IFSs have been popular: one based on a discrete version of the more familiar and classical second order Laplace differential operator of *potential theory*, see [KSW01, Kig04, LNRG96]; and the second approach is based on *Fourier series*, see e.g., [JP98b, DJ06a]. The first model is motivated by infinite discrete network of resistors, and the harmonic functions are defined by minimizing a global measure of resistance, but this approach does not rely on Fourier series. In contrast, the second approach begins with Fourier series, and it has its classical origins in lacunary Fourier series [Kah86].

Hadamard matrices. Let R be a $d \times d$ integer matrix, $B \subset \mathbb{Z}^d$ and $L \subset \mathbb{Z}^d$ having the same cardinality as B, $\#B = \#L =: N$. We call (R, B, L) a *Hadamard triple* if the matrix

$$\frac{1}{\sqrt{N}}(e^{2\pi i R^{-1}b\cdot l})_{b\in B, l\in L} \tag{1.1}$$

is unitary.

Let μ_B be the invariant measure associated to the affine IFS $\tau_b(x) = R^{-1}(x + b)$, $b \in B$. (See Theorem 2.4 below). We conjectured in [DJ07c] that the existence of a set L such that (R, B, L) is a Hadamard triple is sufficient to obtain orthonormal bases of exponentials in $L^2(\mu_B)$.

Conjecture 1.1. *Let R be a $d\times d$ expansive integer matrix, B a subset of $\mathbb{Z}^d$ with $0\in B$. Let μ_B be the invariant measure of the associated IFS $(\tau_b)_{b\in B}$. If there exists a subset L of $\mathbb{Z}^d$ such that (R,B,L) is a Hadamard triple and $0\in L$ then μ_B is a spectral measure.*

In [DJ07c] we proved that this conjecture is true under a certain "reducibility" assumption.

Theorem 1.2. *Let R be an expanding $d\times d$ integer matrix, B a subset of $\mathbb{Z}^d$ with $0\in B$. Assume that there exists a subset L of $\mathbb{Z}^d$ with $0\in L$ such that (R,B,L) is a Hadamard triple which satisfies the reducibility condition (see* [DJ07c]*). Then the invariant measure μ_B is a spectral measure. In particular the conjecture is true in dimension $d=1$.*

In this paper we will show that the conjecture is true also in some other cases, by reducing the problem to some "building blocks" in lower dimensions (Theorem 4.4, Corollary 4.5). We give an example of an affine IFS when the reducibility condition from [DJ07c] is not satisfied but the associated measure is still spectral (Example 5.1).

2. Definitions

The purpose of the present section is to collect the necessary definitions, and to point out the link between the two sides in the general (non-group) Fourier duality; see especially Proposition 2.2 and Definition 2.5, Hadamard-triple.

Definition 2.1. For $\lambda\in\mathbb{R}^d$, let $e_\lambda(x):=e^{2\pi i\lambda\cdot x}$, $x\in\mathbb{R}^d$.

A probability measure on $\mathbb{R}^d$ is called *spectral* if there exists a subset Λ of $\mathbb{R}^d$ such that the family of exponential functions $\{e_\lambda\,|\,\lambda\in\Lambda\}$ is an orthonormal basis for $L^2(\mu)$. In this case Λ is called a spectrum for μ.

The Fourier transform of a measure μ is defined by

$$\hat\mu(x)=\int e^{2\pi i t\cdot x}\,d\mu(t),\quad (x\in\mathbb{R}^d). \tag{2.1}$$

In a number of applications, one encounters a measure μ and a subset Λ such that the functions e_λ indexed by Λ are orthogonal in $L^2(\mu)$, but a separate argument is needed in order to show that the family is total. The following is a universal test which applies to any subset Λ: It is a necessary and sufficient condition on a pair (μ,Λ) allowing us to decide whether μ is spectral with spectrum Λ.

Proposition 2.2. [JP98b, DJ07c] *Let μ be a probability measure on $\mathbb{R}^d$. A subset Λ of $\mathbb{R}^d$ is a spectrum for μ iff*

$$\sum_{\lambda\in\Lambda}|\hat\mu(x+\lambda)|^2=1,\quad (x\in\mathbb{R}^d). \tag{2.2}$$

Remark 2.3. Note that in equation (2.2), we do not have to worry about possible repetitions in Λ. Indeed, if λ appears at least twice in (2.2), then take $x=-\lambda$. Since μ is a probability measure, $\hat\mu(0)=1$. So, the sum on the left of (2.2) is at least 2. This would contradict (2.2), and therefore a λ can appear at most once in the sum.

Affine IFSs. Let R be a $d\times d$ expanding integer matrix, i.e., all eigenvalues λ satisfy $|\lambda|>1$. Let B be a finite subset of $\mathbb{Z}^d$ of cardinality $\#B=:N$, with $0\in B$. We consider the iterated function system

$$\tau_b(x)=R^{-1}(x+b),\quad (x\in\mathbb{R}^d, b\in B). \tag{2.3}$$

Theorem 2.4. [Hut81] *There exists a unique compact set $X_B \subset \mathbb{R}^d$ such that*

$$X_B = \bigcup_{b\in B} \tau_b(X_B). \tag{2.4}$$

Moreover

$$X_B = \left\{ \sum_{k=1}^{\infty} R^{-k} b_k \mid b_k \in B \text{ for all } k \in \mathbb{N} \right\}. \tag{2.5}$$

There exists a unique probability measure $\mu = \mu_{R,B}$ on $\mathbb{R}^d$ such that

$$\int f\, d\mu = \frac{1}{N} \sum_{b\in B} \int f \circ \tau_b \, d\mu, \tag{2.6}$$

for all continuous functions f on $\mathbb{R}^d$.

Hadamard triples.

Definition 2.5. Let R and B as above. Let $L \subset \mathbb{Z}^d$. We say that (R, B, L) form a *Hadamard triple* if $\#L = \#B = N$, $0 \in L$ and the matrix

$$\frac{1}{\sqrt{N}} \left(e^{2\pi i R^{-1} b \cdot l} \right)_{b\in B, l\in L} \tag{2.7}$$

is unitary.

Assumption. *Throughout the paper we will assume that (R, B, L) is a Hadamard triple.*

An easy computation shows that both Definitions 2.1 and 2.5 are closed under taking tensor product. For example, the complex Hadamard matrices occurring in (2.7) include the matrices defining the Fourier transform on finite abelian groups, as well as tensor products of these matrices. Currently there is no complete classification of all the complex Hadamard matrices covered by formula (2.7). Similarly if a fixed complex Hadamard matrix H is given, it is of interest to know all the triples (R, B, L) with the property that H is obtained from (R, B, L) via formula (2.7). Example 5.1 below serves to illustrate these issues.

The purpose of this paper is to explore ways of building higher dimensional spectral pairs from combinatorial algebra applied to lower dimensional systems. Theorems 4.4 and Corollary 4.5 are cases in point. We also address the converse problem of factoring higher dimensional spectral pairs into products of "smaller" ones.

We will denote by S the transpose of R, $S := R^T$. We will need the following "dual" iterated function system

$$\sigma_l(x) = S^{-1}(x + l), \quad (x \in \mathbb{R}^d, l \in L). \tag{2.8}$$

Definition 2.6. Let

$$W_B(x) := \left| \frac{1}{N} \sum_{b\in B} e^{2\pi i b\cdot x} \right|^2, \quad (x \in \mathbb{R}^d). \tag{2.9}$$

Then

$$\sum_{l\in L} W_B(\sigma_l(x)) = 1. \tag{2.10}$$

3. Invariant sets and path measures

The purpose of the present section is to introduce a setting from symbolic dynamics which we will use on a particular given IFS subject to the Hadamard-triple law (Definition 2.5). The data of a Hadamard triple, the scaling matrix R, and the two finite sets B, L allow us to create a digital filter W_B. We then introduce W_B-cycles, and invariant sets. We form a compact space Ω of infinite words in L, and a W_B-path-space measure on Ω, see Definition 3.6. The purpose of this is to allow us to construct candidates for spectra Λ, and then to check the condition in Proposition 2.2.

Invariant sets. For $x \in \mathbb{R}^d$, we call a *trajectory* of x a set of points

$$\{\sigma_{\omega_n} \cdots \sigma_{\omega_1} x \,|\, n \geq 1\}$$

where $\{\omega_n\}_n$ is a sequence of elements in L such that $W_B(\sigma_{\omega_n} \cdots \sigma_{\omega_1} x) \neq 0$ for all $n \geq 1$. We denote by $\mathcal{O}(x)$ the union of all trajectories of x and the closure $\overline{\mathcal{O}(x)}$ is called the *orbit* of x. If $W_B(\sigma_l x) \neq 0$ for some $l \in L$ we say that the *transition* from x to $\sigma_l x$ is possible.

A closed subset $F \subset \mathbb{R}^d$ is called *invariant* if it contains the orbit of all of its points. An invariant subset is called *minimal* if it does not contain any proper invariant subsets. Let

$$\Omega := L^{\mathbb{N}} = \{l_1 l_2 \dots \,|\, l_k \in L, \text{ for all } k \in \mathbb{N}\}$$

the space of infinite words over the alphabet L.

Path measures. For all $x \in \mathbb{R}^d$, there exists a unique probability measure $P_x = P_x(R, B, L)$ on Ω such that for all $l_1, \dots, l_n \in L$, $n \in \mathbb{N}$

$$P_x(\{\omega_1\omega_2 \cdots \in \Omega \,|\, \omega_1 = l_1, \dots, \omega_n = l_n\}) = \prod_{k=1}^{N} W_B(\sigma_k \dots \sigma_1 x). \tag{3.1}$$

The next three propositions can be found in [DJ07c].

Proposition 3.1. *For all $x \in \mathbb{R}^d$*

$$|\hat{\mu}(x)|^2 = \prod_{k=1}^{\infty} W_B(S^{-k} x). \tag{3.2}$$

For all $x \in \mathbb{R}^d$ and all $l_1 l_2 \cdots \in \Omega$

$$P_x(\{l_1 l_2 \dots\}) = \prod_{k=1}^{\infty} W_B(\sigma_{l_k} \dots \sigma_{l_1} x). \tag{3.3}$$

Proposition 3.2. *Let F be a compact invariant subset. Define*

$$N(F) := \{\omega \in \Omega \,|\, \lim_{n\to\infty} d(\sigma_{\omega_n} \cdots \sigma_{\omega_1} x, F) = 0\}. \tag{3.4}$$

(*The definition of $N(F)$ does not depend on x*). *Define*

$$h_F(x) := P_x(N(F)). \tag{3.5}$$

Then $0 \leq h_F(x) \leq 1$, h_F is continuous,

$$\sum_{l \in L} W_B(\sigma_l(x)) h_F(\sigma_l(x)) = h_F(x), \quad (x \in \mathbb{R}^d), \tag{3.6}$$

and for P_x-a.e. $\omega \in \Omega$

$$\lim_{n\to\infty} h_F(\sigma_{\omega_n}\cdots\sigma_{\omega_1}x) = \begin{cases} 1, & \text{if } \omega \in N(F), \\ 0, & \text{if } \omega \notin N(F). \end{cases} \tag{3.7}$$

Proposition 3.3. *Let $F_1, F_2, \dots, F_p$ be a family of mutually disjoint closed invariant subsets of $\mathbb{R}^d$ such that there is no closed invariant set F with $F \cap \bigcup_k F_k = \emptyset$. Then*

$$P_x\left(\bigcup_{k=1}^{p} N(F_k)\right) = 1 \quad (x \in \mathbb{R}^d).$$

Recall that the triples (R, B, L) in Definition 2.5 involve two sets B and L. The role they play is that they are the beginning of a Fourier duality based on the scaling with the matrix R on one side of the duality, and with the transposed matrix $S = R^T$ on the other side. The discussion below is based on this, and the tools we build come from the iterated function system based on the pair (S, L) via formulas (2.8).

Theorem 3.4. [CHR97, Théorème 2.8] *Let K be minimal compact invariant set contained in the set of zeros of an entire function h on $\mathbb{R}^d$.*

a) *There exists V, a proper subspace of $\mathbb{R}^d$ invariant for S (possibly reduced to $\{0\}$), such that K is contained in a finite union $\mathcal{R}$ of translates of V.*
b) *This union contains the translates of V by the elements of a cycle $\{x_0, \sigma_{l_1}x_0, \dots, \sigma_{l_{m-1}}\cdots\sigma_{l_1}x_0\}$ contained in K, and for all x in this cycle, the function h is zero on $x + V$.*
c) *Suppose the hypothesis "(H) modulo V" is satisfied, i.e., for all $p \geq 0$ the equality $\sigma_{k_1}\cdots\sigma_{k_p}0 - \sigma_{k_1'}\cdots\sigma_{k_p'}0 \in V$, with $k_i, k_i' \in L$ implies $k_i - k_i' \in V$ for all $i \in \{1, \dots, p\}$. Then*
$$\mathcal{R} = \{x_0 + V, \sigma_{l_1}x_0 + V, \dots, \sigma_{l_{m-1}}\cdots\sigma_{l_1}x_0 + V\},$$
and every possible transition from a point in $K \cap \sigma_{l_q}\cdots\sigma_{l_1}x_0 + V$ leads to a point in $K \cap \sigma_{l_{q+1}}\cdots\sigma_{l_1}x_0 + V$ for all $1 \leq q \leq m-1$, where $\sigma_{l_m}\cdots\sigma_{l_1}x_0 = x_0$.
d) *Since the function W_B is entire, the union $\mathcal{R}$ is itself invariant.*

A particular example of a minimal compact invariant set is a W_B-cycle. In this case, the subspace V in Theorem 3.4 can be chosen to be $V = \{0\}$:

Definition 3.5. A *cycle* of length m for the IFS $(\sigma_l)_{l\in L}$ is a set of (distinct) points of the form $\mathcal{C} := \{x_0, \sigma_{l_1}x_0, \dots, \sigma_{l_{m-1}}\cdots\sigma_{l_1}x_0\}$, such that $\sigma_{l_m}\cdots\sigma_{l_1}x_0 = x_0$, with $l_1, \dots, l_m \in L$. A *W_B-cycle* is a cycle $\mathcal{C}$ such that $W_B(x) = 1$ for all $x \in \mathcal{C}$.

For a finite sequence $l_1, \dots, l_m \in L$ we will denote by $\underline{l_1 \dots l_m}$ the path in Ω obtained by an infinite repetition of this sequence

$$\underline{l_1 \dots l_m} := (l_1 \dots l_m l_1 \dots l_m \dots)$$

Definition 3.6. A closed invariant set F is called *spectral* if there exists a subset $\Lambda(F)$ of $\mathbb{R}^d$ such that

$$P_x(N(F)) = \sum_{\lambda\in\Lambda(F)} |\hat{\mu}(x+\lambda)|^2, \quad (x \in \mathbb{R}^d). \tag{3.8}$$

In this case $\Lambda(F)$ is called a *spectrum* for F.

Proposition 3.7. *Suppose $(F_i)_{i=1,n}$ are invariant sets with the following properties:*

(i) *The set* F_i *is spectral with spectrum* $\Lambda(F_i)$*;*
(ii) *For each* $1 \leq k \leq n$ *and* $1 \leq i_1 < i_2 < \cdots < i_k \leq n$*, the set* $F_{i_1} \cap F_{i_2} \cap \cdots \cap F_{i_k}$ *is spectral, with spectrum* $\Lambda(F_{i_1}) \cap \cdots \cap \Lambda(F_{i_k})$*.*

Then

$$P_x\left(\bigcup_{i=1}^{n} N(F_i)\right) = \sum_{\lambda \in \cup_{i=1}^{n} \Lambda(F_i)} |\hat{\mu}(x+\lambda)|^2 \tag{3.9}$$

Proof. First we prove that if F_1 and F_2 are invariant sets then

$$N(F_1 \cap F_2) = N(F_1) \cap N(F_2). \tag{3.10}$$

The inclusion "$\subset$" is clear. For the converse, let $\omega \in N(F_1) \cap \ N(F_2)$. Then

$$d(\sigma_{\omega_n} \dots \sigma_{\omega_1}(0), F_i) \to \ 0 \text{ for } i = 1, 2.$$

Suppose, by contradiction, that $d(\sigma_{\omega_n} \dots \sigma_{\omega_1}, F_1 \cap F_2)$ does not converge to 0. Then there exists a $\delta > 0$ and a subsequence such that $y_n := \sigma_{\omega_{k(n)}} \dots \sigma_{\omega_1} 0$ and $d(y_n, F_1 \cap F_2) \geq \delta$. Since 0 is contained in the attractor X_L of the IFS $(\sigma_l)_{l \in L}$, it follows that $y_n \in X_L$ for all n. Since X_L is compact we can find a subsequence such that $\sigma_{\omega_{k(n(p))}} \dots \sigma_{\omega_1} 0$ converges to some point x_0. But then x_0 must be both in F_1 and F_2, which gives the contradiction.

By induction, we can extend (3.10) to any finite number of invariant sets. (Note that the intersection of invariant sets is itself invariant)

We have then

$$P_x\left(\bigcup_{i=1}^{n} N(F_i)\right) = \sum_{k=1}^{n} (-1)^{k+1} \sum_{1 \leq i_1 < \cdots < i_k \leq n} P_x\left(\bigcap_{j=1}^{k} N(F_{i_j})\right) =$$

$$\sum_{k=1}^{n} (-1)^{k+1} \sum_{1 \leq i_1 < \cdots < i_k \leq n} P_x\left(N\left(\bigcap_{j=1}^{k} F_{i_j}\right)\right) = \sum_{k=1}^{n} (-1)^{k+1} \sum_{1 \leq i_1 < \cdots < i_k \leq n} \sum_{\lambda \in \Lambda(\cap_{j=1}^{k} F_{i_j})} |\hat{\mu}(x+\lambda)|^2 =$$

$$\sum_{k=1}^{n} (-1)^{k+1} \sum_{1 \leq i_1 < \cdots < i_k \leq n} \sum_{\lambda \in \cap_{j=1}^{k} \Lambda(F_{i_j})} |\hat{\mu}(x+\lambda)|^2 = \sum_{\lambda \in \cup_{i=1}^{n} \Lambda(F_i)} |\hat{\mu}(x+\lambda)|^2.$$

□

Starting with the iterated function system from the pair (S, L) in formulas (2.8), we have built a system of W_B-cycles $\mathcal{C}$. The proposition below gives conditions for being able to generate a spectrum $\Lambda(\mathcal{C})$ directly from this data.

Proposition 3.8. *Let* $\mathcal{C}$ *be a* W_B*-cycle. Then* $\mathcal{C}$ *is a spectral invariant set with spectrum* $\Lambda(\mathcal{C})$ = *the smallest set* Λ *that contains* $-\mathcal{C}$ *with the property that* $S\Lambda + L \subset \Lambda$.

Proof. Let $\mathcal{C} =: \{x_0, x_1, \dots, x_{m-1}\}$ with $x_k = \sigma_{l_k} x_{k-1}$ for all $k \in \{2, \dots, m\}$ and $\sigma_{l_m} x_{m-1} = x_0$. With [DJ07c, Lemma 4.1], we know that $N(\mathcal{C})$ consists of the infinite words in Ω that end in a repetition of the finite word $l_1 \dots l_m$, i.e., words of the form $\omega_0 \omega_1 \dots \omega_n \underline{l_1 \dots l_m}$.

We proved in [DJ07c, Lemma 4.9] that if $\omega = \omega_0 \dots \omega_{km-1} \underline{l_1 \dots l_m}$, for some $k \geq 0$, $\omega_i \in L$, then

$$P_x(\{\omega\}) = |\hat{\mu}(x + k_{\mathcal{C}}(\omega))|^2, \quad (x \in \mathbb{R}^d)| \tag{3.11}$$

where

$$k_{\mathcal{C}}(\omega) = \omega_0 + S\omega_1 + \dots + S^{km-1}\omega_{km-1} - S^{km} x_0. \tag{3.12}$$

Note that every $\omega \in N(\mathcal{C})$ is of the form $\omega_0 \dots \omega_{km-1}\underline{\tilde{l}_1 \dots \tilde{l}_m}$, where $\tilde{l}_1 \dots \tilde{l}_m$ is a circular permutation of $l_1 \dots l_m$.

Using (3.11), we have to prove only that the set $\Lambda(k_{\mathcal{C}})$ of all numbers $k_{\mathcal{C}}(\omega)$ with ω of the form $\omega_0 \dots \omega_{km-1}\underline{\tilde{l}_1 \dots \tilde{l}_m}$, is the set $\Lambda(\mathcal{C})$ defined in the statement of the Proposition.

Since $\sigma_{l_i} x_{i-1} = x_i$, we have $-x_{i-1} = l_i - Sx_i$ for all i.

First, note that if $\omega = \omega_0 \dots \omega_{km-1}\underline{l_1 \dots l_m}$, and $\omega_{-1} \in L$, then

$$Sk_{\mathcal{C}}(\omega) + \omega_{-1} = \omega_{-1} + S\omega_0 + \dots + \dots + S^{km}\omega_{km-1} - S^{km+1}x_0 =$$

$$\omega_{-1} + \dots + S^{km}\omega_{km} + S^{km+1}l_1 - S^{km+2}x_1 =$$

$$\dots = \omega_{-1} + \dots + S^{km}\omega_{km} + S^{km+1}l_1 + \dots + S^{km+m-1}l_{m-1} - S^{km+m}x_{m-1} =$$

$$k_{\mathcal{C}}(\omega_{-1} \dots \omega_{km-1} l_1 \dots l_{m-1}\underline{l_m l_1 \dots l_{m-1}})$$

This shows that $S\Lambda(k_{\mathcal{C}}) + L \subset \Lambda(k_{\mathcal{C}})$. On the other hand this calculation shows that any $k_C(\omega)$ from the set $\Lambda(k_{\mathcal{C}})$ can be obtained (in a unique way!) from a point in the cycle by applying operations of the form $x \mapsto Sx + l$ with $l \in L$. This implies that $\Lambda(k_{\mathcal{C}}) = \Lambda(\mathcal{C})$.

The uniqueness of the operations $x \mapsto Sx + l$ that lead from a cycle point to a $k_{\mathcal{C}}(\omega)$ comes from the fact that elements in L are incongruent $\bmod S\mathbb{Z}^d$ (see [DJ06b, DJ07c] for more details). $\square$

The next result shows how the spectral problem transforms under coordinate changes.

Definition 3.9. We say that two Hadamard triples (R_1, B_1, L_1) and (R_2, B_2, L_2) are *conjugate* if there exists a matrix $M \in GL_d(\mathbb{Z})$ (i.e., M is invertible, and M and M^{-1} have integer entries) such that $R_2 = MR_1M^{-1}$, $B_2 = MB_1$ and $L_2 = (M^T)^{-1}L_1$.

If the two systems are conjugate then the transition between the IFSs $(\tau_b)_{b\in B_1}$ and $(\tau_{Mb})_{b\in B_1}$ is done by the matrix M; and the transition between the IFSs $(\sigma_l)_{l\in L_1}$ and $(\sigma_{(M^T)^{-1}l})_{l\in L_1}$ is done by the matrix $(M^T)^{-1}$.

Proposition 3.10. *If (R_1, B_1, L_1) and (R_2, B_2, L_2) are conjugate through the matrix M, then*

(i) $\tau_{Mb_1}(Mx) = M\tau_{b_1}(x)$, $\sigma_{(M^T)^{-1}l_1}((M^T)^{-1}x) = (M^T)^{-1}\sigma_{l_1}(x)$, *for all* $b_1 \in B_1$, $l_1 \in L_1$;
(ii) $W_{B_2}((M^T)^{-1}x) = W_{B_1}(x)$ *for all* $x \in \mathbb{R}^d$;
(iii) *For the Fourier transform of the corresponding invariant measures, the following relation holds:* $\hat{\mu}_{B_2}((M^T)^{-1}x) = \hat{\mu}_{B_1}(x)$ *for all* $x \in \mathbb{R}^d$;
(iv) *For* $\omega \in \Omega_1 = L_1^{\mathbb{N}}$, *let* $(M^T)^{-1}\omega := ((M^T)^{-1}\omega_1, (M^T)^{-1}\omega_2, \dots)$. *The associated path measures satisfy the following relation:*

$$P^2_{(M^T)^{-1}x}((M^T)^{-1}E) = P^1_x(E).$$

(v) *If F_1 is invariant for the first IFS, then $(M^T)^{-1}F_1$ is invariant for the second IFS. If in addition F_1 has spectrum $\Lambda(F_1)$, then $(M^T)^{-1}F_1$ has spectrum $(M^T)^{-1}\Lambda(F_1)$.*

Proof. The results follow by direct computation. $\square$

4. Invariant subspaces

In this section we give two ways of building higher dimensional spectral pairs from combinatorial algebra and spectral theory applied to lower dimensional systems. Theorems 4.4 and Corollary 4.5 are cases in point.

Remark 4.1. If V is a *rational* invariant subspace for S, i.e., it has a basis consisting of vectors with rational components then, by [CHR97, Lemme 4.2], there exists a matrix $M \in GL_d(\mathbb{Z})$ as in Proposition 3.10 that maps V into $\mathbb{R}^r \times \{0\}$, with $r = \dim V$. Therefore we can reduce the study to the case when $V = \mathbb{R}^r \times \{0\}$.

Suppose the subspace $\mathbb{R}^r \times \{0\}$ is invariant for S. Then the matrix S has the form

$$S = \begin{bmatrix} S_1 & C \\ 0 & S_2 \end{bmatrix} \tag{4.1}$$

where S_1 is $r \times r$, S_2 is $(d-r) \times (d-r)$, C is $(d-r) \times r$ and 0 is $r \times (d-r)$.
The matrix R has the form:

$$R = \begin{bmatrix} A_1 & 0 \\ C^* & A_2 \end{bmatrix}, \text{ and } R^{-1} = \begin{bmatrix} A_1^{-1} & 0 \\ -A_2^{-1}C^*A_1^{-1} & A_2^{-1} \end{bmatrix}, \tag{4.2}$$

with $A_1 = S_1^T$, $A_2 = S_2^T$, $C^* = C^T$. By induction,

$$R^{-k} = \begin{bmatrix} A_1^{-k} & 0 \\ D_k & A_2^{-k} \end{bmatrix}, \text{ where } D_k := -\sum_{l=0}^{k-1} A_2^{-(l+1)} C^* A_1^{-(k-l)}. \tag{4.3}$$

The set B can be written as

$$B = \{(r_i, \eta_{i,j}) \mid i \in \{1, \dots, N_1\}, j \in \{1, \dots, N_2(i)\}\} \tag{4.4}$$

with $\{r_1, \dots, r_{N_1}\} = \operatorname{proj}_{\mathbb{R}^r} B$.
We have

$$X_B = \{\sum_{k=1}^{\infty} R^{-k} b_k \mid b_k \in B\}.$$

Therefore any element (x, y) in X_B can be written in the following form:

$$x = \sum_{k=1}^{\infty} A_1^{-k} r_{i_k}, \quad y = \sum_{k=1}^{\infty} D_k r_{i_k} + \sum_{k=1}^{\infty} A_2^{-k} \eta_{i_k, j_k}.$$

Define

$$X_1 := \{\sum_{k=1}^{\infty} A_1^{-k} r_{i_k} \mid i_k \in \{1, \dots, N_1\}\}.$$

Let μ_1 be the invariant measure for the iterated function system

$$\tau_{r_i}(x) = A_1^{-1}(x + r_i), \quad i \in \{1, \dots, N_1\}.$$

The set X_1 is the attractor of this iterated function system.
Assumption. *We will assume that the measure μ_1 has no overlap, i.e., $\mu_1(\tau_{r_i}(X_1) \cap \tau_{r_j}(X_1)) = 0$ for $i \neq j$.*
We will also use the contractions

$$\sigma_l(x) = S_2^{-1}(x + l), \quad (x, l \in \mathbb{R}^{d-r}).$$

For each sequence $\omega = (i_1 i_2 \dots) \in \{1, \dots, N_1\}^{\mathbb{N}} =: \Omega_1$, define $x(\omega) = \sum_{k=1}^{\infty} A_1^{-k} r_{i_k}$. Also, because of the non-overlap condition, for μ_1-a.e. $x \in X_1$, there is a *unique* ω such that $x(\omega) = x$. We define this as $\omega(x)$. This establishes an a.e. bijective correspondence between Ω_1 and X_1, $\omega \leftrightarrow x(\omega)$.

For $\omega = (i_1 i_2 \dots) \in \Omega_1$ define

$$\Omega_2(\omega) := \{\eta_{i_1,j_1}\eta_{i_2,j_2}\dots\eta_{i_n,j_n}\dots \,|\, j_k \in \{1, \dots, N_2(i_k)\}\}.$$

For $\omega \in \Omega_1$

$$g(\omega) := \sum_{k=1}^{\infty} D_k r_{i_k}, \text{ and } g(x) := g(\omega(x)). \tag{4.5}$$

Also we denote $\Omega_2(x) := \Omega_2(\omega(x))$.

For $x \in X_1$, define

$$X_2(x) := X_2(\omega(x)) := \left\{ \sum_{k=1}^{\infty} A_2^{-k} \eta_{i_k,j_k} \,\middle|\, j_k \in \{1, \dots, N_2(i_k)\} \text{ for all } k \right\}.$$

Note that the attractor X_B has the following form:

$$X_B = \{(x, g(x) + y) \,|\, x \in X_1, y \in X_2(x)\}.$$

We will show that the measure μ can also be decomposed as a product between the measure μ_1 and some measures μ^2_ω on $X_2(\omega)$.

On $\Omega_2(\omega)$, consider the product probability measure $\mu(\omega)$ which assigns to each η_{i_k,j_k} equal probabilities $1/N_2(i_k)$.

Next we define the measure μ^2_ω on $X_2(\omega)$. Let $r_\omega : \Omega_2(\omega) \to X_2(\omega)$,

$$r_\omega(\eta_{i_1,j_1}\eta_{i_2,j_2}\dots) = \sum_{k=1}^{\infty} A_2^{-k} \eta_{i_k,j_k}.$$

Define the measure $\mu^2_x := \mu^2_{\omega(x)} := \mu\omega(x) \circ r^{-1}_{\omega(x)}$.

In building higher dimensional spectral pairs from lower dimensional systems by factoring it turns out that overlap properties in the lower dimensional systems play a critical role. This is made precise in the next Proposition. The proof is contained in [DJ07c]; note that the more restrictive assumptions used there are not needed for this Proposition.

Proposition 4.2. *Using the notations above, suppose*

(a) *The measure μ_1 has no overlap, i.e., $\mu_1(\tau_{r_i}(X_1) \cap \tau_{r_j}(X_1)) = 0$ for all $i \neq j$.*
(b) *The numbers $N_2(i) = \{\eta \,|\, (r_i, \eta) \in B\}$ are all the same, $N_2(i) = N_2$, $i \in \{1, \dots, N_1\}$. In particular $N_1 N_2 = N$.*

Let σ be the shift on Ω_1, $\sigma(i_1 i_2 \dots) = (i_2 i_3 \dots)$. Let $\omega = (i_1 i_2 \dots) \in \Omega_1$.

(i) *For all measurable sets E in $X_2(\omega)$,*

$$\mu^2_\omega(E) = \frac{1}{N_2} \sum_{j=1}^{N_2(i_1)} \mu^2_{\sigma(\omega)}(\tau^{-1}_{\eta_{i_1,j}}(E)).$$

The Fourier transform of the measure μ^2_ω satisfies the equation:

$$\hat{\mu}^2_\omega(y) = m(S_2^{-1} y, i_1)\hat{\mu}^2_{\sigma(\omega)}(S_2^{-1} y), \tag{4.6}$$

where

$$m(y, i_1) = \frac{1}{N_2} \sum_{j=1}^{N_2} e^{2\pi i \eta_{i_1,j} \cdot y}.$$

(ii) *For all continuous functions on* $\mathbb{R}^d$*:*

$$\int_{X_B} f \, d\mu = \int_{X_1} \int_{X_2(x)} f(x, y + g(x)) \, d\mu_x^2(y) \, d\mu_1(x).$$

(iii) *If* Λ_1 *is a spectrum for the measure* μ_1*, then*

$$F(y) := \sum_{\lambda_1 \in \Lambda_1} |\hat{\mu}(x + \lambda_1, y)|^2 = \int_{X_1} |\hat{\mu}_s^2(y)|^2 \, d\mu_1(s) \quad (x \in \mathbb{R}^r, y \in \mathbb{R}^{d-r}).$$

Moreover, if

$$\tilde{W}(y) := \frac{1}{N_1} \sum_{i=1}^{N_1} |m(y, i)|^2, \quad (y \in \mathbb{R}^{d-r}), \tag{4.7}$$

then

$$F(y) = \tilde{W}(S_2^{-1} y) F(S_2^{-1} y), \text{ and } F(y) = \prod_{k=1}^{\infty} \tilde{W}(S_2^{-k} y), \quad (y \in \mathbb{R}^{d-r}). \tag{4.8}$$

Definition 4.3. We say that $p \in \mathbb{R}^d$ is an S-*period* for a function f on $\mathbb{R}^d$ if

$$f(x + S^n p) = f(x), \text{ for all } x \in \mathbb{R}^d, n \geq 0.$$

Theorem 4.4. *Suppose* $y_0 \in \mathbb{R}^{d-r}$ *and the set* $\mathbb{R}^r \times \{y_0\}$ *is invariant. Then* $\mathbb{R}^r \times \{0\}$ *is invariant for* S*. Using the notations above, there exists* $(l_1^0, l_2^0) \in L$ *such that* $\sigma_{l_2^0}(y_0) = y_0$*.*

Assume that the conditions (a) and (b) in Proposition 4.2 are satisfied. Assume in addition that the following conditions are satisfied:

(i) *The measure* μ_1 *is spectral with spectrum* Λ_1*.*

(ii) *The spectrum* Λ_1 *has the following properties:*

$$S(\Lambda \times \{-y_0\}) + L \supset \Lambda \times \{-y_0\}. \tag{4.9}$$

and

$$(\lambda_1, -y_0) \text{ is an } S\text{-period for } W_B, \quad (\lambda_1 \in \Lambda_1). \tag{4.10}$$

(iii) $(A_1, \{r_i \,|\, i \in \{1, \dots, N_1\}, L_1(l_2^0) := \{l_1 \,|\, (l_1, l_2^0) \in L\})$ *is a Hadamard triple.*

Then $\mathbb{R}^r \times \{y_0\}$ *is a spectral set with spectrum*

$$\Lambda := \bigcup_{n \in \mathbb{N}} (L + SL + \cdots + S^{n-1} L + S^n(\Lambda_1 \times \{-y_0\})). \tag{4.11}$$

Proof. Take $x \in \mathbb{R}^r$, then from (2.10) we deduce that there exists $l = (l_1^0, l_2^0) \in L$ such that $W_B(\sigma_l(x, y_0)) \neq 0$. Then for $x' \in \mathbb{R}^r$ small enough, we have that $W_B(\sigma_l(x + x', y_0)) \neq 0$. Therefore the transition $(x + x', y_0) \mapsto \sigma_l(x + x', y_0)$ is possible. But, since $\mathbb{R}^r \times \{y_0\}$ is invariant, it follows that $\mathbb{R}^r \times \{y_0\} \ni \sigma_l(x + x', y_0) = S^{-1}(x + x', y_0)$. Subtracting, we get $S^{-1}(x', 0) \in \mathbb{R}^r \times \{0\}$. Then we can multiply by scalars to see that $S^{-1}(x, 0) \in \mathbb{R}^r \times \{0\}$ for all $x \in \mathbb{R}^r$, so $\mathbb{R}^r \times \{0\}$ is invariant for S.

Then we have

$$\sigma_l(x, y_0) = S^{-1}(x + l_1^0, y_0 + l_2^0) = (S_1^{-1}(x + l_1^0) + D_1^T(y_0 + l_2^0), S_2^{-1}(y_0 + l_2^0)) \in \mathbb{R}^r \times \{y_0\}$$

which implies that $\sigma_{l_2^0}(y_0) = y_0$.

Next, we compute for $(x, y) \in \mathbb{R}^d$:

$$\sum_{l_1 \in L_1(l_2^0)} W_B(\sigma_{(l_1, l_2^0)}(x, y)) = \sum_{l_1} \frac{1}{N^2} \sum_{i,i'=1}^{N_1} \sum_{j,j'=1}^{N_2} e^{2\pi i (r_i - r'_i)\cdot(S_1^{-1}(x+l_1)+D_1^T(y+l_2^0)) + (\eta_{i,j}-\eta_{i',j'})\cdot(S_2^{-1}(y+l_2^0))}$$

$$= (*)$$

But using the Hadamard property in (iii) we have

$$\frac{1}{N_1} \sum_{l_1 \in L_1(l_2^0)} e^{2\pi i (r_i - r'_i)\cdot S_1^{-1} l_1} = \begin{cases} 1, & i = i' \\ 0, & i \neq i'. \end{cases}$$

So

$$(*) = \frac{1}{N_1 N_2^2} \sum_i \sum_{j,j'} e^{2\pi i (\eta_{i,j} - \eta_{i',j'})\cdot S_2^{-1}(y+l_2^0)} = \frac{1}{N_1} \sum_{i=1}^{N_1} |m(\sigma_{l_2^0} y, i)|^2 = \tilde{W}(\sigma_{l_2^0}(y)).$$

We obtain the equation

$$\sum_{l_1 \in L_1(l_2^0)} W_B(\sigma_{(l_1, l_2^0)}(x, y)) = \tilde{W}(\sigma_{l_2^0}(y)), \quad (y \in \mathbb{R}^{d-r}). \tag{4.12}$$

Using [DJ07c, Lemma 4.1], we have that $N(\mathbb{R}^r \times \{y_0\})$ consists of all the infinite words in $L^{\mathbb{N}}$ for which the second components are l_2^0 from some point on.

We show first that

$$P_{(x,y)}(\{(\omega_1 \dots \omega_n \dots) \,|\, \omega_{n,2} = l_2^0 \text{ for all } n\}) = \prod_{k=1}^{\infty} \tilde{W}(\underbrace{\sigma_{l_2^0} \cdots \sigma_{l_2^0}}_{k \text{ times}} y). \tag{4.13}$$

We compute for all n, by summing over all the possibilities for the first component, and using (3.3):

$$P_{(x,y)}(\{(\omega_1\omega_2 \dots) \,|\, \omega_{k,2} = l_2^0, 1 \leq k \leq n\}) = \sum_{l_{1,1},\dots,l_{n,1}} \prod_{k=1}^{n} W_B(\sigma_{(l_{k,1}, l_2^0)} \cdots \sigma_{(l_{1,1}, l_2^0)}(x, y)) = (*).$$

Using (4.12) we obtain further

$$(*) = \tilde{W}(\sigma_{l_2^0} \cdots \sigma_{l_2^0} y) \sum_{l_{1,1},\dots,l_{n-1,1}} \prod_{k=1}^{n-1} W_B(\sigma_{(l_{k,1}, l_2^0)} \cdots \sigma_{(l_{1,1}, l_2^0)}(x, y)) = \dots$$

$$= \prod_{k=1}^{n} \tilde{W}(\underbrace{\sigma_{l_2^0} \cdots \sigma_{l_2^0}}_{k \text{ times}} y).$$

Then, letting $n \to \infty$ we obtain (4.13).

We have

$$\sigma_{l_2^0}(y) = S_2^{-1}(y + l_2^0) = S_2^{-1}(y - y_0) + S_2^{-1}(y_0 + l_2^0) = S_2^{-1}(y - y_0) + y_0.$$

$$\sigma_{l_2^0}\sigma_{l_2^0}(y) = S_2^{-2}(y - y_0) + S_2^{-1}(y_0 + l_2^0) = S_2^{-2}(y - y_0) + y_0.$$

By induction

$$\underbrace{\sigma_{l_2^0}\cdots\sigma_{l_2^0}}_{k\text{ times}} = S_2^{-k}(y-y_0)+y_0, \quad (k\geq 1) \tag{4.14}$$

Therefore

$$P_{(x,y)}(\{(\omega_1\dots\omega_n\dots)\,|\,\omega_{n,2}=l_2^0 \text{ for all } n\}) = \prod_{k=1}^{\infty}\tilde{W}(S_2^{-k}(y-y_0)+y_0). \tag{4.15}$$

Since $\mathbb{R}^r\times\{y_0\}$ is invariant, we have $W_B(\sigma_l(x,y_0))=0$ for all $x\in\mathbb{R}^r$ and $l=(l_1,l_2)\in L$ with $l_2\neq l_2^0$. This implies that

$$\tilde{W}(y_0)=\tilde{W}(\sigma_{l_2^0}y_0) \overset{\text{by (4.12)}}{=} \sum_{l_1\in L_1(l_2^0)} W_B(\sigma_{(l_1,l_2^0)}(0,y_0)) = \sum_{l\in L} W_B(\sigma_l(0,y_0)) \overset{\text{by (2.10)}}{=} 1.$$

But then, with (4.7), $|m(y_0,i)|=1$ for all $i\in\{1,\dots,N_1\}$. Using the formula for $m(y_0,i)$ we get that for each $i\in\{1,\dots,N_1\}$ the numbers $e^{2\pi i\eta_{i,j}}$, $j\in\{1,\dots,N_2\}$ are all the same (otherwise $|m(y_0,i)|<1$). And this shows that $|m(y+y_0,i)|=|m(y,i)|$ so $\tilde{W}(y+y_0)=\tilde{W}(y)$ for all $y\in\mathbb{R}^{d-r}$.

Plug this into (4.15), we get using Proposition 4.2(iii) and the notations there:

$$\begin{aligned} P_{(x,y)}(\{(\omega_1\dots\omega_n\dots)\,|\,\omega_{n,2}=l_2^0 \text{ for all } n\}) &= \prod_{k=1}^{\infty}\tilde{W}(S_2^{-k}(y-y_0)) = F(y-y_0) \\ &= \sum_{\lambda_1\in\Lambda_1}|\hat{\mu}(x+\lambda_1,y-y_0)|^2. \end{aligned} \tag{4.16}$$

Next, fix $l_1,\dots,l_n\in L$. We compute, using (3.3) and (4.15)

$$\begin{aligned} &P_{(x,y)}(\{\omega\,|\,\omega_1=l_1,\dots,\omega_n=l_n,\omega_{k,2}=l_2^0 \text{ for } k\geq n+1\}) = \\ &W_B(\sigma_{l_1}(x,y))\dots W_B(\sigma_{l_n}\dots\sigma_{l_1}(x,y))P_{\sigma_{l_n}\dots\sigma_{l_1}(x,y)}(\{\omega\,|\,\omega_{k,2}=l_2^0, \text{ for all } k\geq 1\}) = \\ &W_B(\sigma_{l_1}(x,y))\dots W_B(\sigma_{l_n}\dots\sigma_{l_1}(x,y))\sum_{\lambda_1\in\Lambda_1}|\hat{\mu}(\sigma_{l_n}\dots\sigma_{l_1}(x,y)+(\lambda_1,-y_0))|^2. \end{aligned} \tag{4.17}$$

But

$$\begin{aligned} W_B(\sigma_l(x,y))|\hat{\mu}(\sigma_l(x,y)+(\lambda_1,-y_0))|^2 &= W_B(S^{-1}((x,y)+l))|\hat{\mu}(S^{-1}((x,y)+l+S(\lambda_1,-y_0)))|^2 \\ &\overset{\text{by (4.10)}}{=} W_B(S^{-1}((x,y)+l+S(\lambda_1,-y_0)))|\hat{\mu}(S^{-1}((x,y)+l+S(\lambda_1,-y_0)))|^2 \\ &= |\hat{\mu}((x,y)+l+S(\lambda_1,-y_0))|^2. \end{aligned}$$

By induction,

$$\begin{aligned} &W_B(\sigma_{l_1}(x,y))\dots W_B(\sigma_{l_n}\dots\sigma_{l_1}(x,y))|\hat{\mu}(\sigma_{l_n}\dots\sigma_{l_1}(x,y)+(\lambda_1,-y_0))|^2 = \\ &\qquad |\hat{\mu}((x,y)+l_n+Sl_{n-1}+\dots+S^{n-1}l_1+S^n(\lambda_1,-y_0))|^2. \end{aligned}$$

Using this in (4.17), we get

$$\begin{aligned} &P_{(x,y)}(\{\omega\,|\,\omega_1\in L,\dots,\omega_n\in L,\omega_{k,2}=l_{2,0} \text{ for } k\geq n+1\}) = \\ &\qquad \sum_{l_1,\dots,l_n\in L,\lambda_1\in\Lambda}|\hat{\mu}((x,y)+l_n+Sl_{n-1}+\dots+S^{n-1}l_1+S^n(\lambda_1,-y_0))|^2. \end{aligned} \tag{4.18}$$

We know by [DJ07c, Lemma 4.1], that $N(\mathbb{R}^r \times \{y_0\})$ consists of all the infinite words in $L^{\mathbb{N}}$ for which the second component are l_2^0 from some point on.

By (4.9), the sets $L+SL+\cdots+S^{n-1}L+S^n(\Lambda_1\times\{-y_0\})$ are increasing with n. Therefore

$$P_{(x,y)}(N(\mathbb{R}^r\times\{y_0\})) = \sum_{\lambda\in\cup_n(L+SL+\cdots+S^{n-1}L+S^n(\Lambda_1\times\{-y_0\}))} |\hat{\mu}(x+\lambda)|^2.$$

□

In the next result we combine the conditions from the theorem into a form that can be used in checking concrete examples.

Corollary 4.5. *Suppose the set $\mathbb{R}^r \times \{0\}$ is invariant. Then it is also invariant for the matrix S. Let $L_1(0) = \{l_1 \,|\, (l_1, 0) \in L\}$. Assume condition (a) in Proposition 4.2 is satisfied, and in addition, the following hold:*

(i) *The measure μ_1 is spectral with spectrum Λ_1.*
(ii) *$S_1\Lambda_1 + L_1(0) \supset \Lambda_1$, and every λ_1 in Λ_1 is an S_1-period for*

$$W_1(x) := \left|\frac{1}{N_1}\sum_{i=1}^{N_1} e^{2\pi i r_i\cdot x}\right| \quad (x\in\mathbb{R}^r).$$

Then $\mathbb{R}^r\times\{0\}$ is a spectral invariant set, with spectrum

$$\Lambda(\mathbb{R}^r\times\{0\}) := \bigcup_{n\geq 0}(L+SL+\cdots+S^{n-1}L+S^n(\Lambda_1\times\{0\})).$$

Proof. We use Theorem 4.4. We get that $\mathbb{R}^r\times\{0\}$ is invariant for S, and $l_2^0 = 0$ in our case. We have to check that all conditions are satisfied. Since $\mathbb{R}^r\times\{0\}$ is an invariant set, we have that $W_B(\sigma_l(x,0)) = 0$ for all $l\in L$ with $l_2\neq 0$. This implies, using (2.10) that

$$1 = \sum_{l_1\in L_1(0)} W_B(\sigma_{(l_1,0)}(x,0)) = \sum_{l_1\in L_1(0)} W_B(S_1^{-1}(x+l_1),0) =$$

$$\sum_{l_1\in L_1(0)}\frac{1}{N^2}\sum_{i,i'=1}^{N_1}\sum_{j=1}^{N_2(i)}\sum_{j'=1}^{N_2(i')} e^{2\pi i(r_i-r_i')\cdot(S_1^{-1}(x+l_1))} = \sum_{l_1\in L_1(0)}\sum_{i,i'=1}^{N_1}\frac{N_2(i)}{N}\frac{N_2(i')}{N}e^{2\pi i(r_i-r_i')\cdot(S_1^{-1}(x+l_1))}$$

$$= \sum_{l_1\in L_1(0)}\left|\sum_{i=1}^{N_1}\frac{N_2(i)}{N}e^{2\pi i r_i\cdot(S_1^{-1}(x+l_1))}\right|^2 = \sum_{l_1\in L_1(0)}|\hat{\delta}(S_1^{-1}(x+l_1))|^2,$$

where δ is the discrete measure $\delta = \sum_{i=1}^{N_1}\frac{N_2(i)}{N}\delta_{r_i}$, and $\hat{\delta}$ is its Fourier transform. Since $\sum_i N_2(i) = N$, the measure δ is a probability measure.

Then, the previous calculation and Proposition 2.2 implies that $S_1^{-1}L_1(0)$ is a spectrum for the measure δ. With [DJ07c, Lemma 2.7] we obtain that the numbers $N_2(i)/N$ are all equal. This shows that condition (b) in Proposition 4.2 is satisfied. Also, since $S_1^{-1}L_1(0)$ is a spectrum for δ this implies that $\#L_1(0) = N_1$ and $(A_1, \{r_1,\dots,r_{N_1}\}, L_1(0))$ is a Hadamard triple.

Finally, for $\lambda_1\in\Lambda_1$ we have $S^n(\lambda_1,0) = (S_1^n\lambda_1, 0)$ so $(\lambda_1,0)$ is an S-period for W_B.

Thus all the conditions of Theorem 4.4 are satisfied and the result follows. □

5. Examples

The following example is a spectral pair in $\mathbb{R}^2$ whose Hadamard triple is not reducible to $\mathbb{R}\times\{0\}$, i.e., to the first coordinate. The Hadamard matrix of the system is of the form $H\otimes H$ where H is the 2×2 unitary matrix of the Fourier transform on $\mathbb{Z}_2$, the cyclic group of order 2.

Example 5.1. We give an example of an affine IFS that satisfies the Hadamard condition, but not the reducibility condition of [DJ07c, Theorem 3.8].

Let

$$R:=\begin{bmatrix}4&0\\0&4\end{bmatrix}, B:=\left\{\begin{bmatrix}0\\0\end{bmatrix},\begin{bmatrix}0\\2\end{bmatrix},\begin{bmatrix}1\\4\end{bmatrix},\begin{bmatrix}1\\6\end{bmatrix}\right\}, L:=\left\{\begin{bmatrix}0\\0\end{bmatrix},\begin{bmatrix}2\\0\end{bmatrix},\begin{bmatrix}2\\1\end{bmatrix},\begin{bmatrix}0\\5\end{bmatrix}\right\}$$

.

Then the matrix in (2.7) is

$$\frac{1}{\sqrt{4}}\begin{bmatrix}1&1&1&1\\1&1&-1&-1\\1&-1&-1&1\\1&-1&1&-1\end{bmatrix}$$

which is unitary so (R,B,L) is a Hadamard pair.

The subspace $\mathbb{R}\times\{0\}$ is invariant. Indeed, we have

$$W_B(x,y)=\left|\frac{1}{4}(1+e^{2\pi i2y}+e^{2\pi i(x+4y)}+e^{2\pi i(x+6y)})\right|^2=\left|\frac{1}{4}(1+e^{2\pi i2y})(1+e^{2\pi i(x+4y)})\right|^2.$$

And

$$\sigma_{(0,0)}(x,0)=(\frac{x}{4},0),\sigma_{(2,0)}(x,0)=(\frac{x+2}{4},0),\sigma_{(2,1)}(x,0)=(\frac{x+2}{4},\frac{1}{4}),\sigma_{(0,5)}(x,0)=(\frac{x}{4},\frac{5}{4}),$$

so $W_B(\sigma_{(2,1)}(x,0))=W_B(\sigma_{(0,5)}(x,0))=0$, for all $x\in\mathbb{R}$.

However the Hadamard triple is not reducible to $\mathbb{R}\times 0$ because the set L does not satisfy condition [DJ07c, Definition 3.1(iii)]: the number of vectors in L that have the second component 0 is 2, and there is only one vector that has the second component 1.

Theorem 5.2. *The measure $\mu=\mu_B$ is spectral.*

Proof. We use Theorem 3.4, and look for candidates for subspaces V such that some finite union $\mathcal{R}=\cup_{i=1}^n(x_i,y_i)+V$ is invariant and contains some minimal invariant set K. Note that since K is minimal, every orbit of every point in K is equal to K. Thus we can take (x_i,y_i) to be in K and also be limit points of some trajectory. So we can assume $(x_i,y_i)\in X_L$, the attractor of the IFS $(\sigma_l)_{l\in L}$.

Using (2.5), we have that $X_L\subset[0,\frac{2}{3}]\times[0,\frac{5}{3}]$.

Since $\mathcal{R}$ is invariant we must have for all $(x,y)\in(x_i,y_i)+V$ and $l\in L$ either $\sigma_l(x,y)\in\mathcal{R}$ or $W_B(\sigma_l(x,y))=0$. Since X_L is not contained in any finite union of translates of a subspace, we must have $W_B(\sigma_l(x,y))=0$ for some $(x,y)\in\mathcal{R}$ and some $l\in L$.

Looking at the formula for W_B we see that $W_B(x',y')=0$ iff $4y'=2k+1$ for some $k\in\mathbb{Z}$ or $2x'+8y'=2k+1$ for some $k\in\mathbb{Z}$. Thus, the subspaces we are looking for are $V_1=\{(x,y)\,|\,y=0\}=\mathbb{R}\times\{0\}$ and $V_2=\{(x,y)\,|\,x+4y=0\}$.

We analyze each subspace separately, and compute what are the possible unions $\mathcal{R}$.

For $V_1 = \mathbb{R} \times \{0\}$: since we have $W_B(\sigma_l(x_i, y_i)) = 0$ for some $l = (l_1, l_2)$ this implies that $4((y_i + l_2)/4)$ is of the form $2k + 1$ for some $k \in \mathbb{Z}$. So $y_i \in \mathbb{Z}$. And since $(x_i, y_i) \in X_L \subset [0, 2/3] \times [0, 5/3]$ this implies that $y_i = 0$ or $y_i = 1$.

For $y_i = 0$ we obtain the invariant set $\mathbb{R}\times\{0\}$. For $y_i = 1$, since we have $\sigma_{(2,1)}(*, 1) = (*, \frac{1}{2})$ and $W_B(*, \frac{1}{2}) \neq 0$, it follows that $(*, \frac{1}{2}) \in \mathcal{R}$. By induction $(*, \frac{1}{2\cdot 4^n}) \in \mathcal{R}$, which contradicts the fact that $\mathcal{R}$ is a *finite* union of translates of $\mathbb{R} \times \{0\}$.

Then for $V_2 = \{(x, y) \,|\, x+4y = 0\}$ we will use a matrix M to conjugate our IFS to another one for which V_2 becomes $\mathbb{R} \times \{0\}$, and we use Proposition 3.10. Take $M = \begin{bmatrix} 4 & -1 \\ 0 & 1 \end{bmatrix}$. Then $\tilde{R} = MRM^{-1} = R$,

$$\tilde{B} = MB = B := \left\{ \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \begin{bmatrix} -2 \\ 1 \end{bmatrix} \right\},$$

$$\tilde{L} := (M^T)^{-1}L = \left\{ \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 2 \end{bmatrix}, \begin{bmatrix} -1 \\ 6 \end{bmatrix}, \begin{bmatrix} -5 \\ 20 \end{bmatrix} \right\}.$$

$$W_{\tilde{B}}(x, y) = \frac{1}{4}(1 + e^{2\pi i(-2x)})(1 + e^{2\pi i y}).$$

The subspace $V_2 = \{(4t, -t) \,|\, t \in \mathbb{R}\}$ is mapped into $\mathbb{R} \times \{0\}$ by $(M^T)^{-1}$. We look for possible invariant unions $\mathcal{R}$ of the form $\cup_i (x_i, y_i) + \mathbb{R} \times \{0\}$.

We note that $X_{\tilde{L}} \subset [-\frac{5}{3}, 0]\times[0, \frac{20}{3}]$. Since $\sigma_l(*, y_i) = 0$ for some $l \in \tilde{L}$ we get $2\cdot\frac{y_i+l_2}{4} = 2k+1$ for some $k \in \mathbb{Z}$. Therefore y_i must be an even integer in $[0, \frac{20}{3}]$. Thus $y_i \in \{0, 2, 4, 6\}$.

We will use the following notation: if $(*, a)$ is in $\tilde{R}$ and for some $l = (l_1, l_2)$ we have that $2 \cdot \frac{a+l_2}{4}$ is not an odd integer, then $W_B(\sigma_l(*, a)) \neq 0$ so $(*, \frac{a+l_2}{4})$ must be in $\mathcal{R}$; we write $a \xrightarrow{l_2} \frac{a+l_2}{4}$.

If $y_i = 0$, then $0 \xrightarrow{20} \frac{20}{4} = 5 \xrightarrow{0} \frac{5}{4} \xrightarrow{0} \frac{5}{16} \xrightarrow{0} \dots$, and this contradicts the finiteness of the union $\mathcal{R}$. If $y_i = 2$, then $2 \xrightarrow{2} \frac{2+2}{4} = 1 \xrightarrow{0} \frac{1}{4} \xrightarrow{0} \dots$. If $y_i = 4$ then $4 \xrightarrow{0} 1 \xrightarrow{0} \frac{1}{4} \xrightarrow{0} \dots$. If $y_i = 6$ then $6 \xrightarrow{2} 2 \xrightarrow{2} 1 \xrightarrow{0} \frac{1}{4} \dots$. In all cases we obtain a contradiction. Therefore V_2 does not produce invariant sets correlated to minimal invariant sets K as in Theorem 3.4 (one needs to apply the conjugation matrix back to (R, B, L)).

Thus, the only invariant sets we have to worry about are $\mathbb{R} \times \{0\}$ and possible W_B-cycles.

First, let us see what the spectrum is for $\mathbb{R} \times \{0\}$. We use Corollary 4.5. We have $N_1 = 2$, $N_2 = 2$, $\{r_1, r_2\} = \{0, 1\}$. The measure μ_1 is associated to the IFS $\tau_0(x) = \frac{x}{4}$, $\tau_1 x = \frac{x+1}{4}$, so it clearly has no overlap. The measure μ_1 is a spectral measure with spectrum

$$\Lambda_1 := \left\{ \sum_{k=0}^{n} 4^k a_k \,|\, a_k \in \{0, 2\} \right\}.$$

(This is just a rescale of the first example of a fractal spectral measure given by Jorgensen and Pedersen in [JP98b]). Since $\Lambda_1 \subset \mathbb{Z}$ we see that condition (ii) in Corollary 4.5 is satisfied. Hence $\mathbb{R} \times \{0\}$ has spectrum

$$\Lambda(\mathbb{R} \times \{0\}) = \bigcup_{n\geq 0} (L + SL + \dots S^{n-1}L + S^n(\Lambda_1 \times \{0\})).$$

It remains to look for W_B-cycles. By [DJ07b, Theorem 4.1] we have to compute the lattice $\Gamma = \{\gamma \in \mathbb{R}^2 \,|\, \gamma \cdot b \in \mathbb{Z} \text{ for all } b \in B\}$. So if $(x, y) \in \Gamma$ then $2x, x + 4y, x + 6y \in \mathbb{Z}$ so $2y \in \mathbb{Z}$

and $x \in \mathbb{Z}$, which means that $\Gamma = \mathbb{Z} \times \frac{1}{2}\mathbb{Z}$. To find the W_B-cycles we intersect Γ with X_L. Since $X_L \subset [0, \frac{2}{3}] \times [0, \frac{5}{3}]$ it follows that the only candidates for W_B-cycle points are $(0,0), (0, \frac{1}{2}), (0,1)$.

$(0,0)$ is the trivial cycle, but we can discard it because it is contained in $\mathbb{R} \times \{0\}$. It is easy to check that the other two are not W_B-cycles.

We conclude that the invariant set $\mathbb{R} \times \{0\}$ contains all minimal invariant sets, and therefore, by Proposition 3.3, the spectrum of μ is $\Lambda(\mathbb{R} \times \{0\})$. □

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